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Monochromatic Trees with Respect to Edge Partitions*

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It is shown, that for every infinite cardinal κ there exists a graph F on κ vertices satisfying $F \rightarrow (T)_{\lambda}^{\text{edges}}$ for every tree T on κ vertices and all λ satisfying $\operatorname{cf}\kappa \rightarrow (\omega)_{\lambda}^{3}$. © 1993 Academic Press, Inc.

1. PARTITIONING EDGES

Erdős, Hajnal, and Pósa asked, whether for every (infinite) graph G there exists a graph F, such that for every two-coloring of the edges of F there is some monochromatic and induced copy of G in F. For finite (resp. countable) graphs G the answer is known to be affirmative, cf., [2, 3, 7].

With respect to uncountable graphs the results turn out to depend very much on the underlying set theoretical assumptions, compare [8, 9].

In this paper we investigate the particular case, when G is a tree on κ vertices. Our results do not use set theoretic assumptions and in most cases they give the smallest possible size of the Ramsey graphs.

Notations. For graphs F and H we denote by $\binom{F}{H}$ the set of all induced subgraphs of F which are isomorphic to H. "$F \rightarrow (G)_{\lambda}^{H}$," where λ is some cardinal, abbreviates the following statement: *for every λ-coloring $\Delta\colon \binom{F}{H} \rightarrow \lambda$ there exists some induced G-subgraph $G^* \in \binom{F}{G}$ such that $\Delta \mid \binom{G^*}{H}$ is a constant mapping*. In particular, if H is the complete graph on two

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vertices (i.e., H is a single edge), we write $F \to (G)^e_\lambda$. We say that H is weak subgraph of H if a subgraph which is not necessarily induced. The weak Ramsey-arrow "$F \to (G)^{H}_\lambda$" is defined as above, however, just referring to weak subgraphs. If H is the complete graph on two vertices (i.e., H is a single edge), we write $F \to (G)^e_\lambda$. For infinite cardinals κ, the κ-regular tree, i.e., every vertex has degree κ, is denoted by T_κ.

The main result of our paper is the following:

THEOREM A. *For every infinite cardinal κ there exists a graph G_κ on κ vertices, such that $G_\kappa \to (T_\kappa)^e_\lambda$ holds for all cardinals λ satisfying $cf\kappa \to (\omega)^3_\lambda$.*

From Ramsey's theorem we have the following corollary:

COROLLARY A1. *For every infinite cardinal κ there exists a graph G_κ on κ vertices, such that $G_\kappa \to (T_\kappa)^e_n$ holds for every positive integer n.*

This improves the result of [6], where it has been shown that there exists a graph F on $\exp(\kappa)$ vertices satisfying $F \to (T_\kappa)^e_n$ for every positive integer. As by [4], $(2^{2^\lambda})^+ \to (\omega)^3_\lambda$ for every infinite cardinal λ; hence we have another corollary:

COROLLARY A2. *For every infinite cardinal λ there exists a graph G_κ on $\kappa = (2^{2^\lambda})^+$ vertices satisfying $G_\kappa \to (T_\kappa)^e_\lambda$.*

Proof of Theorem A. Let κ be an infinite cardinal. The shift graph $G_\kappa = (V_\kappa, E_\kappa)$ on κ is defined as follows: $V_\kappa = [\kappa]^2$, $E_\kappa = \{\{A, B\} \in [V_\kappa]^2 \mid \max A = \min B\}$. Thus every three-element subset $\{x, y, z\} \in [\kappa]^3$, say $x < y < z$, corresponds to the edge $\{\{x, y\}, \{y, z\}\} \in E_\kappa$, and vice versa. We shall show that G_κ has the desired properties.

CLAIM 1. *Let $W \subseteq V_\kappa$ be a set of vertices and let $E^* \subseteq E_\kappa \cap [W]^2$ be a set of edges of G_κ, such that the graph (W, E^*) has minimal degree κ. Then (W, E^*) contains an induced copy of T_κ, which also is an induced subgraph with respect to G_κ.*

Proof of Claim 1. Using transfinite recursion, we shall show that there exists a one-to-one mapping φ: $V(T_\kappa) \to W$ such that for all $x, y \in V(T_\kappa)$ holds (i) $\{x, y\} \in E(T_\kappa)$ implies that $\{\varphi(x), \varphi(y)\} \in E^*$ and (ii) $\{x, y\} \notin E(T_\kappa)$ implies that $\{\varphi(x), \varphi(y)\} \notin E_\kappa$. Then $\{\varphi(x) \mid x \in V(T_\kappa)\}$ is the desired copy of T_κ.

Let $\leqslant$ be a well-ordering of $V(T_\kappa)$ having order type κ and let $r \in V(T_\kappa)$ be its least element. Put $R_0 = \{r\}$ and define φ_0: $R_0 \to W$ arbitrarily.

Assume, by induction, that for every $\nu<\xi<\kappa$ sets $R_\nu \subseteq V(T_\kappa)$ and mappings $\varphi_\nu\colon R_\nu \to W$ have been defined such that

(1) $R_\nu \subseteq R_\mu$ and $\varphi_\mu \rceil R_\nu = \varphi_\nu$ for all $\nu<\mu<\xi$,

(2) for all $\nu<\mu<\xi$ and all $x \in R_\nu$ and $y \in R_\mu$ we have:

(2i) $\{x, y\} \in E(T_\kappa)$ implies that $\{\varphi_\nu(x), \varphi_\mu(y)\} \in E^*$,

(2ii) $\{x, y\} \notin E(T_\kappa)$ implies that $\{\varphi_\nu(x), \varphi_\mu(y)\} \notin E_\kappa$,

(2iii) $\max \varphi_\nu(x) \neq \max \varphi_\mu(y)$,

(3) every R_ν is a connected subset of vertices of T_κ and $|R_\nu| \leqslant |\nu| + 1$.

Let $z \in V(T_\kappa)$ be the least element (w.r.t. $\leqslant$) which is not already contained in $\bigcup_{\nu<\xi} R_\nu$, but which is connected by an edge in $E(T_\kappa)$ to some (uniquely determined, as T_κ is a tree) vertex $x \in \bigcup_{\nu<\xi} R_\nu$. Say $x \in R_\mu$ and $\varphi_\mu(x) = \{\alpha, \beta\}$. Since $|\bigcup_{\nu<\xi} R_\nu| \leqslant |\xi| + 1 < \kappa$ and as, with respect to E^*, vertex $\{\alpha, \beta\} \in W$ has degree κ, there exists some $\gamma<\kappa$ such that $\{\{\alpha, \beta\}\{\beta, \gamma\}\} \in E^*$, but $\gamma \notin \varphi_\nu(y)$ for all $\nu<\xi$ and $y \in R_\nu$.

Put $R_\xi = \{z\} \cup \bigcup_{\nu<\xi} R_\nu$ and $\varphi_\xi(z) = \{\beta, \gamma\}$ and $\varphi_\xi(y) = \varphi_\nu(y)$ for all $\nu<\xi$ and $y \in R_\nu$. Then again properties (1), (2), and (3) are satisfied, and, by construction, $\varphi = \bigcup_{\nu<\kappa} \varphi_\nu$ has the desired properties. ▮

Claim 2. *Let λ satisfy $cf\kappa \to (\omega)^3_\lambda$ and let $\Delta\colon E_\kappa \to \lambda$ be a mapping. Then there exists a κ-subset $Y \in [\kappa]^\kappa$ and there exists some $\eta<\lambda$, such that the graph (Y, E^{**}), where $E^{**} = \{\{A, B\} \in [Y]^2 \cap E_\kappa \mid \Delta(\{A, B\}) = \eta\}$, is κ-regular.*

Proof of Claim 2. Assume to the contrary that Claim 2 fails to be true. Then, by transfinite induction, one easily defines for every $\xi<\lambda$ a well-ordering $<_\eta$ of $[\kappa]^2$ with the following property:

$$|\{B \in [\kappa]^2 \mid A <_\xi B, \{A, B\} \in E_\kappa \text{ and } \Delta(\{A, B\}) = \eta\}| < \kappa$$
$$\text{for every } A \in [\kappa]^2. \tag{$*$}$$

Consider the following recursive construction: Put $\alpha_0 = 0$ and $R_0 = \kappa$ and assume, by induction, that for every $\nu < \xi < \mathrm{cf}\kappa$ ordinals $\alpha_\nu < \kappa$ and subsets $R_\nu \subseteq \kappa$ have been defined such that (i) $|\kappa \setminus R_\nu| < \kappa$ and (ii) $\alpha_\nu = \min \bigcap_{\mu<\nu} R_\mu$ for every $\nu<\xi$.

Put $R'_\xi = \bigcap_{\nu<\xi} (R_\nu \setminus \{\alpha_\nu\})$. Note that $|\kappa \setminus R'_\xi| < \kappa$, as $\xi < \mathrm{cf}\kappa$. Put $\alpha_\xi = \min R'_\xi$. For every ordinal $\nu<\xi$ consider the vertex $A = \{\alpha_\nu, \alpha_\xi\}$ and remove from R'_ξ all ordinals β with $A <_\eta \{\alpha_\xi, \beta\}$, $\{A, \{\alpha_\xi, \beta\}\} \in E_\kappa$, and $\Delta(\{A, \{\alpha_\xi, \beta\}\}) = \eta$ for some $\eta<\lambda$. According to $(*)$, the remaining subset, call it R_ξ, satisfies (i), and, moreover,

$$\{\alpha_\xi, \gamma\} <_\eta \{\alpha_\nu, \alpha_\xi\} \qquad \text{for every } \gamma \in R_\xi \text{ and every } \nu<\xi$$
$$\text{with } \Delta(\{\{\alpha_\nu, \alpha_\xi\}, \{\alpha_\xi, \gamma\}\}) = \eta. \tag{$**$}$$

Consider the set $\mathscr{A} = \{\alpha_\xi \mid \xi < \mathrm{cf}\,\kappa\}$. As $\mathrm{cf}\,\kappa \to (\omega)^3_\lambda$, there exists an infinite subset $\{\beta_0, \beta_1, ...\} \subseteq \mathscr{A}$, say, $\beta_0 < \beta_1 < \cdots$, and there exists some $\eta < \lambda$ such that

$$\Delta(\{\{\beta_i, \beta_{i+1}\}, \{\beta_{i+1}, \beta_{i+2}\}\}) = \eta \qquad \text{for every} \quad i < \omega.$$

By (**), this implies that

$$\{\beta_{i+1}, \beta_{i+2}\} <_\eta \{\beta_i, \beta_{i+1}\} \qquad \text{for every} \quad i < \omega,$$

contradicting that $<_\xi$ is a well-ordering. Thus the claim is proved. ▮

Finally, Theorem A is an immediate conclusion drawn from Claim 2 and Claim 1. ▮

Remark 1. For later use let us note that a straightforward modification of the proof of Claim 1 establishes the following lemma:

LEMMA 1. *Let $G = (V, E)$ be a graph of minimal degree κ, where κ is an infinite cardinal. Then G contains a (non-induced) subgraph isomorphic to T_κ.*

Proof. Again, let $\leqslant$ be a well-ordering of $V(T_\kappa)$ having order type κ and let r be its minimal element. Put $R_0 = \{r\}$ and define $\varphi_0 \colon R_0 \to V$ arbitrarily. Assume that for every $\nu < \xi < \kappa$ connected subsets $R_\nu \subseteq V(T_\kappa)$ and embeddings $\varphi_\nu \colon R_\nu \to V$ have been defined such that $R_\nu \subseteq R_\mu$ and $\varphi_\mu \upharpoonright R_\nu = \varphi_\nu$ for all $\nu < \mu < \xi$. Let $z \in V(T_\kappa)$ be the least element of $V(T_\kappa) \setminus \bigcup_{\nu<\xi} R_\nu$ which is connected by an edge to some $x \in \bigcup_{\nu<\xi} R_\nu$. Say $x \in R_\mu$ and $\varphi_\mu(x) = a$. As a is joined by an edge to at least κ many vertices in V, there exists some $b \in V \setminus \bigcup_{\nu<\xi} \varphi_\nu(R_\nu)$ such that $\{a, b\} \in E$. Put $R_\xi = \{z\} \cup \bigcup_{\nu<\xi} R_\nu$ and define $\varphi_\xi(z) = b$ and $\varphi_\xi(y) = \varphi_\nu(y)$ for every $y \in R_\nu$, $\nu < \xi$. By construction then $\varphi = \bigcup_{\nu<\kappa} \varphi_\nu$ is a weak embedding of T_κ into G. ▮

With a different proof, Lemma 1 can be found in [1].

Remark 2. The condition "$\mathrm{cf}\,\kappa \to (\omega)^3_\lambda$" cannot be weakened to "$\kappa \to (\omega)^3_\lambda$." This can be seen, e.g., by considering $K_{\aleph_\omega}$, the complete graph on $\aleph_\omega$ vertices:

LEMMA 2. *Consider $K_{\aleph_\omega} = (\aleph_\omega, [\aleph_\omega]^2)$, the complete graph on $\aleph_\omega$ vertices. Then $K_{\aleph_\omega} \stackrel{\text{weak}}{\not\longrightarrow} (T_{\aleph_\omega})^e_\omega$; i.e., there exists a mapping $\Delta \colon [\aleph_\omega]^2 \to \omega$ such that no (weak) subgraph $(\aleph_\omega, \Delta^{-1}(\xi))$, $\xi < \omega$, contains some $\aleph_\omega$-regular (weak) subgraph, where "weak" means "not necessarily induced."*

Proof. Put $\Delta(\{\alpha, \beta\}) = (i, k)$ for all $\aleph_{i-1} \leqslant \alpha < \aleph_i$ and $\aleph_{k-1} \leqslant \beta < \aleph_k$, where $\aleph_{-1} = 0$ for convenience. ▮

Remark 3. Corollary A2 suggests considering the function f, which is defined for infinite cardinals λ such that $f(\lambda)$ is the least cardinal κ for which there exists a graph G on κ vertices with $G \to (T_\kappa)^e_\lambda$. From Corollary A2 it follows that $f(\lambda) \leqslant (2^{2^\lambda})^+$. On the other hand, we can show that $\exp(\lambda)^+ \leqslant f(\lambda)$.

LEMMA 3. *For* $\kappa = \exp(\lambda)$ *the complete graph on* κ *vertices,* K_κ, *satisfies* $K_\kappa \not\xrightarrow{\text{weak}} (T_\kappa)^e_\lambda$; *i.e., there exists a mapping* Δ: $[\exp(\lambda)]^2 \to \lambda$ *such that no (weak) subgraph* $(\exp(\lambda), \Delta^{-1}(\xi))$, $\xi < \lambda$, *contains some* $\exp(\lambda)$*-regular subgraph.*

Proof. Let us view $\exp(\lambda)$ as the set of 0–1 sequences of length λ. Let $\leqslant^*$ be a well-ordering of type $\exp(\lambda)$ of these sequences. The mapping Δ is defined as follows:

$$\begin{aligned}\Delta(\{(x_i)_{i<\lambda}, (y_i)_{i<\lambda}\}) &= (\xi, 0) \qquad \text{if} \quad \min\{i<\lambda \mid x_i \neq y_i\} = \xi \\ &\qquad\qquad \text{and } x_\xi = 0 \text{ and } (x_i)_{i<\lambda} <^* (y_i)_{i<\lambda} \\ &= (\xi, 1) \qquad \text{if} \quad \min\{i<\lambda \mid x_i \neq y_i\} = \xi \\ &\qquad\qquad \text{and } x_\xi = 0 \text{ and } (x_i)_{i<\lambda} {}^*\!> (y_i)_{i<\lambda}.\end{aligned}$$

Intuitively, we did the following: The complete graph $[\exp(\lambda)]^2$ is partitioned into λ many bipartite graphs, where $\{(x)_{i<\lambda}, (y)_{i<\lambda}\}$ belongs to the ξth bipartite graph iff $\xi = \min\{i \mid x_i \neq y_i\}$. For each such bipartite graph we have a left-hand side (those $(x_i)_{i<\lambda}$ with $x_\xi = 0$) and a right-hand side (those $(x_i)_{i<\lambda}$ with $x_\xi = 1$). Now the edge $\{(x_i)_{i<\lambda}, (y_i)_{i<\lambda}\}$ belongs to class $(\xi, 0)$ iff $(x_i)_{i<\lambda} <^* (y_i)_{i<\lambda}$. ∎

We conjecture that $f(\lambda) = \exp(\lambda)^+$. But we can only show the following:

LEMMA 4. $K_{\exp(\lambda)^+} \xrightarrow{\text{weak}} (T_{\exp(\lambda)^+})^e_\lambda$ *for every infinite cardinal* λ.

Proof. In view of Lemma 1 it suffices to prove the following claim:

CLAIM 3. *For every mapping* Δ: $[\exp(\lambda)^+]^2 \to \lambda$ *there exists a subset* $Y \in [\exp(\lambda)^+]^{\exp(\lambda)^+}$ *and there exists some* $\eta < \lambda$, *such that for every* $\alpha \in Y$ *there exists* $Z_\alpha \in [Y]^{\exp(\lambda)^+}$ *with* $\Delta(\{\alpha, \beta\}) = \eta$ *for every* $\beta \in Z_\alpha$.

Proof of Claim 3. We use a slight modification of the proof of Claim 2. Assume to the contrary, that the claim fails to be true. Then for every $\xi < \lambda$ there exists a well-ordering $<_\eta$ of $\exp(\lambda)^+$ such that

$$\begin{aligned}&|\{\beta \in \exp(\lambda)^+ \mid \alpha <_\eta \beta \text{ and } \Delta(\{\alpha, \beta\}) = \eta\}| \\ &\qquad < \exp(\lambda)^+ \qquad \text{for every} \quad \alpha \in \exp(\lambda).\end{aligned} \tag{$*$}$$

Let $\xi < (\exp\lambda)^+$ and assume by induction that for every $\nu < \xi$ ordinals α_ν and sets $R_\nu \subseteq (\exp\lambda)^+$ have been defined such that (i) $|(\exp\lambda)^+ - R_\nu| < (\exp\lambda)^+$ and $\alpha_\xi = \min \bigcap_{\nu<\xi} R_\nu$ for every $\nu < \xi$. Put $\alpha_\xi = \min \bigcap_{\nu<\xi} R_\nu$, where, for convenience, $\bigcap_\varnothing R_\nu = (\exp\lambda)^+$. For every ordinal $\nu \leqslant \xi$ remove from $\bigcap_{\nu<\xi} R_\nu$ all ordinals β with $\alpha_\nu <_\eta \beta$ and $\Delta(\{\alpha_\nu, \beta\}) = \eta$. According to $(*)$ and (i), the remaining set, call it R_ξ, again satisfies (i). Moreover,

$$\beta <_\eta \alpha_\nu \text{ for every } \nu \leqslant \xi \text{ and every } \beta \in R_\xi \text{ with } \Delta(\{\alpha_\nu, \beta\}) = \eta. \qquad (**)$$

Consider the set $\mathscr{A} = \{\alpha_\nu \mid \nu < (\exp\lambda)^+\}$. As $(\exp\lambda)^+ \rightarrow (\omega)^2_\lambda$ [4], there exists an infinite subset $\{\beta_0, \beta_1, ...\} \subseteq \mathscr{A}$, say $\beta_0 < \beta_1 < \cdots$, and there exists $\eta < \lambda$ such that $\Delta(\{\beta_i, \beta_{i+1}\}) = \xi$ for every $i < \omega$. By $(**)$ this implies that $\beta_{i+1} <_\eta \beta_i$ for every $i < \omega$, contradicting that $<_\eta$ is a well-ordering. ∎

2. Ramifications

1. *Partitioning Vertices*

If H is a single vertex, we write $F \rightarrow (G)^v_\lambda$ instead of $F \rightarrow (G)^H_\lambda$. Using the same method as for Theorem A we can show:

THEOREM B. *For every infinite cardinal κ there exists a graph G_κ on κ vertices, such that $G_\kappa \rightarrow (T_\kappa)^v_\lambda$ for all satisfying $cf\kappa \rightarrow (\omega)^2_\lambda$.*

Proof of Theorem B. We use the same graph G_κ as for Theorem A. Let $\Delta\colon [\kappa]^2 \rightarrow \lambda$ be an arbitrary mapping. This can be viewed as an edge-coloring of the complete graph K_κ. Hence, by Lemma 4, there exists a weakly embedded copy of T_κ in K_κ, say with edges $W \subseteq [\kappa]^2$, which is colored monochromatically. Viewing these edges as vertices of G_κ, shows that $(W, [W]^2 \cap E_\kappa)$ has minimal degree κ. Hence, by Claim 1, Theorem B follows. ∎

Remark. Theorem B suggests considering the function $g(\lambda)$ which is defined for infinite cardinals such that $g(\lambda)$ is the minimal cardinal κ such that there exists a graph G on κ vertices satisfying $G \rightarrow (T_\kappa)^v_\lambda$. Clearly, $\lambda^+ \leqslant g(\lambda)$, and from Theorem B we infer that $g(\lambda) \leqslant (\exp\lambda)^+$. Again, we conjecture that the lower bound gives the right value.

3. Directed Graphs

Let us denote by $\vec{T}_\kappa$ the directed κ-regular arborescence; i.e., call some distinguished vertex $r \in V(T_\kappa)$ the root of $\vec{T}_\kappa$ and direct all edges of $E(T_\kappa)$ in such a way that every vertex $x \in V(T_\kappa) \setminus \{r\}$ can be reached from r by a directed path. By $\overleftarrow{T}_\kappa$ we denote the directed tree obtained from $\vec{T}_\kappa$ by reversing the orientation.

The directed version of Theorem A then says:

THEOREM C. *For every infinite cardinal κ there exists a directed graph $\vec{G}_\kappa$ on κ vertices, such that $\vec{G} \to (\vec{T}_\kappa)^e_\lambda$ holds for all cardinals λ satisfying $cf\kappa \to (\omega)^3_\lambda$.*

COROLLARY C1. *For every infinite cardinal κ there exists a directed graph $\vec{G}_\kappa$ on κ vertices, such that $\vec{G}_\kappa \to (\vec{T}_\kappa)^e_n$ holds for every positive integer n.*

COROLLARY C2. *For every infinite cardinal κ there exists an undirected graph G_κ on κ vertices, such that for every orientation $\vec{G}^*_\kappa$* of *G_κ, $\vec{G}^*_\kappa$ contains some induced copy of $\vec{T}_\kappa$ or it contains some induced copy of $\overleftarrow{T}_\kappa$.*

Proof of Corollary C2. Let G_κ be the undirected version of the graph $\vec{G}_\kappa$ from Corollary C1. Given an arbitrary orientation $\vec{G}^*_\kappa$, impose a two-coloring on the edges of $\vec{G}_\kappa$ by comparing the two orientations and apply Corollary 1. ▮

Corollaries C1 and C2 improve results of [6], where the existence of such graphs $\vec{G}_\kappa$ on $\exp(\kappa)$ vertices has been established.

COROLLARY C3. *For every infinite cardinal λ there exists a directed graph $\vec{G}_\kappa$ on κ vertices, where $\kappa = \exp_2(\lambda)^+$, such that $\vec{G}_\kappa \to (\vec{T}_\kappa)^e_\lambda$.*

Proof of Theorem C. Let G_κ be the graph from Theorem A. Direct each edge $\{\{\alpha, \beta\}, \{\beta, \gamma\}\}$ from $\{\alpha, \beta\}$ to $\{\beta, \gamma\}$. Then the proof of Theorem A actually establishes Theorem C. ▮

REFERENCES

1. L. D. ANDERSON AND C. THOMASSEN, The cover index of finite graphs, *Aequationes Math.* **20** (1980), 244–251.
2. W. DEUBER, A Generalization of Ramsey's theorem, *in* "Infinite and Finite Sets," Vol. I (A. Hajnal, R. Rado, and V. T. Sós, Eds.), Colloq. Math. Soc. Bolyai, Vol. 10, pp. 323–332, North-Holland, Amsterdam, 1975.
3. P. ERDŐS, A. HAJNAL, AND L. PÓSA, Strong embeddings of graphs into colored graphs, *in* "Infinite and Finite Sets" (A. Hajnal, R. Rado, and V. Sós, Eds.), Colloq. Math. János Bolyai, Vol. 10, pp. 584–596, North-Holland, Amsterdam, 1975.
4. P. ERDŐS AND R. RADO, A partition calculus in set theory, *Bull. Amer. Math. Soc.* **62** (1950), 427–489.
5. R. L. GRAHAM, B. L. ROTHSCHILD, AND J. H. SPENCER, "Ramsey Theory," Wiley, New York, 1980.

6. H. J. PRÖMEL, C. THOMASSEN, AND B. VOIGT, Induced oriented trees in graphs, unpublished manuscript, Bielefeld, 1980.
7. V. RÖDL, "Generalization of Ramsey's Theorem and Dimension of Graphs," Charles University Prague, Thesis 1973 [Czech]; English translation, Generalization of Ramsey's theorem, *in* "Graphs Hypergraphs and Block Systems," pp. 211–220, Zielona Gora, 1976.
8. A. HAJNAL AND P. KOMJÁTH, Embedding graphs into colored graphs, *Trans. Amer. Math. Soc.* **307** (1988), 395–409.
9. S. SHELAH, Consistency of positive partition theorems for graphs and models, *in* "Set Theory and Its Applications" (J. Steprāns and S. Watson, Eds.), pp. 167–193, Springer-Verlag, New York/Berlin, 1989.